E 2.5 Signals & Linear Systems

Tutorial sheet 8 Solutions

1) Given that bandwidth of $f(t)$ is 10kHz,
sampling frequency $F_s \geq 2 \times 10\text{kHz}$
$\geq 20{,}000$.

If we have frequency resolution ~~at~~ $f_0 = 50\text{Hz}$,
the time window T_ω required to ~~for~~ provide
the DFT is $T_\omega = \frac{1}{50\text{Hz}} = 20\text{ms}$.

$$\therefore \; N_0 \geq \frac{1/F_s}{T_\omega} \geq 400.$$

Since N_0 must be a power of 2,
choose $N_0 = 512$. //

Now if we have 512 samples at $T_s = 50\mu s$
we need a signal of duration
$512 \times 50\mu s = 25.6\text{ms}$.

Since we only have a signal duration
of 10ms, we need to zero padding
over 15.6ms //

②

2) $f(t) = e^{-t}u(t) \iff F(j\omega) = \frac{1}{j\omega + 1}$

$$|F(\omega)| = \frac{1}{\sqrt{\omega^2+1}} \simeq \frac{1}{\omega} \quad \text{for } \omega >> 1.$$

From results of sheet 7 Q4, we know that the essential bandwidth of $f(t)$ is $10.13a$ Hz $(a=1)$ ~~or 2π10.13 Hz~~ or ~ 10 Hz.

$\therefore$ Choose $F_s = 20$ Hz & Sampling interval $T_s = 0.05$ sec.

We assume that $e^{-t}u(t)$ becomes negligible after, say, 6 time constants ($e^{-6} \simeq 0$).

Choose time window $T_0 = 6$ sec.

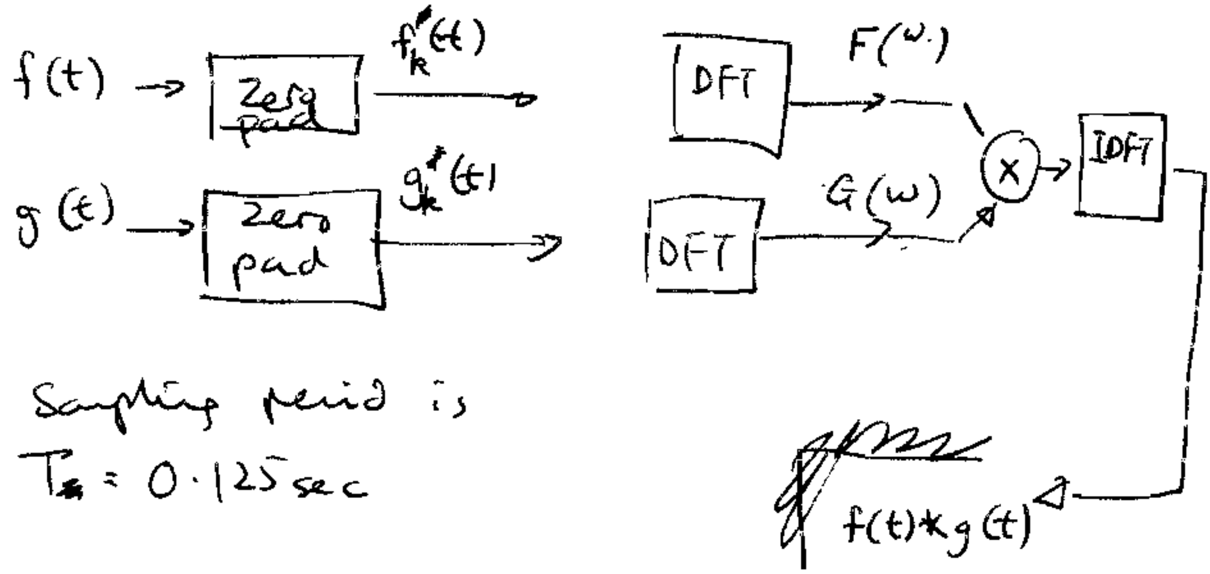
$N_0 = \frac{6}{0.05} = 120.$ Take $N_0 = 128$.

$T = 0.05 \quad T_0 = 128 \times 0.05 = 6.4$.

$\therefore F_0$ (frequency resolution) $= \frac{1}{6.4} = 0.15625$ Hz

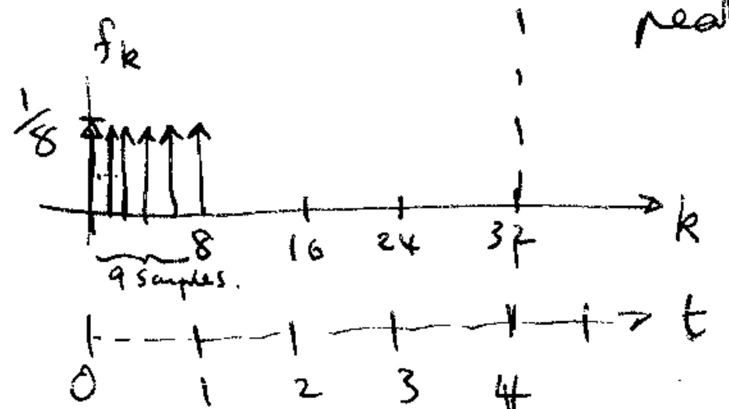
3) The widths of f(t) & g(t) are 1 and 2 sec respectively. Hence the width of the convolved signal is 3 sec.

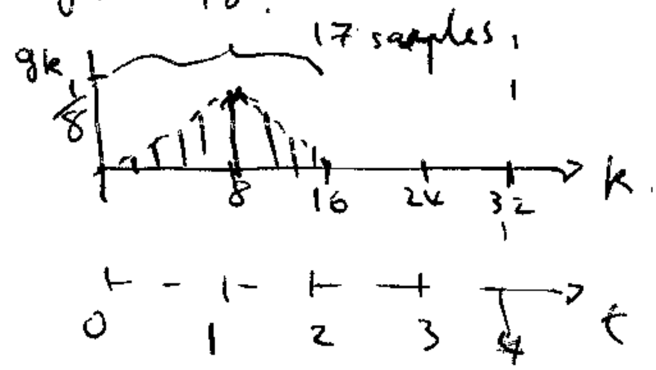
∴ We need to zero-pad f(t) & g(t) to make up 3 sec, before performing fast convolution via DFT.



Sampling period is
$T = 0.125$ sec

$\therefore N_0 = \frac{3}{0.125} = 24$.
Make $N_0 = 32$. ∴ Adjust signal to 4 sec long.
Observe that $T_0/N_0 = \frac{1}{8}$ ∴ Adjust digital samples with scaling 1/8. ∴ $f_k = 1/8$ & peak $g_k = 1/8$.





4) (a) $F[z] = \sum_{k=1}^{\infty} \gamma^{k-1} z^{-k}$

$$= \frac{1}{\gamma} \sum_{k=1}^{\infty} \left(\frac{\gamma}{z}\right)^{k}$$

$$= \frac{1}{\gamma} \left[\frac{\gamma}{z} + \left(\frac{\gamma}{z}\right)^2 + \left(\frac{\gamma}{z}\right)^3 + \cdots \right]$$

$$= \frac{1}{\gamma} \left[-1 + \left(1 + \frac{\gamma}{z} + \left(\frac{\gamma}{z}\right)^2 + \cdots \right) \right]$$

$$= \frac{1}{\gamma} \left[-1 + \frac{1}{1 - \frac{\gamma}{z}} \right] = \frac{1}{z - \gamma} \quad //$$

(b) $F[z] = \sum_{k=m}^{\infty} z^{-k} = z^{-m} + z^{-(m+1)} + z^{-(m+2)} + \cdots$

$$= z^{-m} \left[1 + z^{-1} + z^{-2} + \cdots \right]$$

$$= z^{-m} \left(\frac{1}{1 - \frac{1}{z}} \right) = \frac{z}{z^m (z-1)} \quad //$$

(c) $F[z] = \sum_{k=0}^{\infty} \frac{\gamma^k}{k!} z^{-k} = \sum_{k=0}^{\infty} \frac{1}{k!} \left(\frac{\gamma}{z}\right)^k$

Since $e^x = \sum_{k=0}^{\infty} \frac{1}{k!} x^k$

$\therefore F[z] = e^{\frac{\gamma}{z}} \quad //$

5) (a)

$$f[k] = 2^{k+1}u[k-1] + e^{k-1}u[k]$$

$$= \underbrace{4 \times 2^{k-1}u[k-1]}_{\Updownarrow \atop \frac{4}{z-2}} + \underbrace{\frac{1}{e}e^{k}u[k]}_{\Updownarrow \atop \frac{1}{e}\frac{z}{z-e}}.$$

$$\therefore F[z] = \frac{4}{z-2} + \frac{1}{e}\frac{z}{z-e}$$ //

(b)

$$f[k] = k\gamma^{k}u[k-1]$$

$$= k\gamma^{k}(u[k] - \delta[k])$$

$$= k\gamma^{k}u[k] - 0$$

$$= k\gamma^{k}u[k].$$

$$\therefore F[z] = \frac{\gamma z}{(z-\gamma)^2}$$ //

(c)

$$f[k] = \left[2^{-k}\cos\frac{\pi}{3}k\right]u[k-1]$$

$$= 2^{-k}\cos\frac{\pi}{3}k\left\{u[k] - \delta[k]\right\}$$

$$= \cancel{2^{-k}\cos\frac{\pi}{3}k}$$

$$\therefore F(z) = \frac{z(z-0.25)}{z^2-0.5z+0.25} - 1$$

$$= \frac{0.25(z-1)}{z^2-0.5z+0.25}$$ //

6) (a) $\frac{F(z)}{z} = \frac{z-4}{(z-2)(z-3)}$

$= \frac{2}{z-2} - \frac{1}{z-3}$

$\therefore F(z) = \frac{2z}{z-2} - \frac{z}{z-3}$

$\therefore f[k] = [2 \times 2^k - 3^k]\, u[k]$ //

(b) $\frac{F[z]}{z} = \frac{e^{-2}-2}{(z-e^{-2})(z-2)}$

$= \frac{1}{z-e^{-2}} - \frac{1}{z-2}$.

$F[z] = \frac{z}{z-e^{-2}} - \frac{z}{z-2}$

$f[k] = [e^{-2k} - 2^k]\, u[k]$ //

(c) $\frac{F(z)}{z} = \frac{-5z+22}{(z+1)(z-2)^2}$

$= \frac{3}{z+1} + \frac{k}{z-2} + \frac{4}{(z-2)^2}$.

Multiply both sides by z & let $z \to \infty$ yields $0 = 3 + k + 0 \Rightarrow k = -3$

$\therefore F[z] = 3\frac{z}{z+1} - 3\frac{z}{z-2} + 4\frac{z}{(z-2)^2}$

$f[k] = \{ 3(-1)^k - 3(2)^k + 2k(2)^k]\, u[k]$ //

6) (d) $$\frac{F(z)}{z} = \frac{2z^2 - 0.3z + 0.25}{z(z^2+0.6z+0.25)}$$
$$= \frac{1}{z} + \frac{Az + B}{z^2+0.6z+0.25}$$

Multiply both sides by z & let $z \to \infty$.
This yields
$$2 = 1 + A \Rightarrow A = 1.$$

Setting $z = 1$ on both sides yields
$$\frac{1.95}{1.85} = 1 + \frac{1+B}{1.85}$$
$$\Rightarrow B = -0.9.$$

$$\therefore F[z] = 1 + \frac{z(z-0.9)}{z^2+0.6z+0.25}$$

For the second term on the right side, we use pair #12c in z-transform table:
$$A = 1, B = -0.9, a = 0.3, |\gamma| = 0.5.$$
$$r = \sqrt{10} \qquad \beta = \cos^{-1}\left(\frac{-0.3}{0.5}\right) = 2.214$$
$$\theta = \tan^{-1} \frac{1.2}{0.4} = 1.249.$$

$$\therefore f[k] = \delta[k] + \sqrt{10}\, 0.5^k \cos(2.214k + 1.249)\, u[k]$$

7) Use long division yields:

$$\frac{2z^3+13z^2+z}{z^3+7z^2+2z+1} = 2 - \frac{1}{z} + \frac{4}{z^2} + \cdots$$

Therefore $f[0]=2$, $f[1]=-1$, $f[2]=4$ //

8). $f[k] = u[k] - u[k-m]$.

$$\therefore F[z] = \frac{z}{z-1} - z^{-m}\left(\frac{z}{z-1}\right)$$

$$= \frac{1-z^{-m}}{1-z^{-1}}$$ //.